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Remainders of arcwise connected compactifications of the plane

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Abstract

A. Lelek asked which continua are remainders of locally connected compactifications of the plane. In this paper we study a similar problem with local connectedness replaced by arcwise connectedness. (Each locally connected continuum is arcwise connected.) We give the following characterization: a continuum X is pointed 1-movable if and only if there is an arcwise connected compactification of the plane with X as the remainder.

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1. Introduction

Every space considered in this paper is metric and every map is continuous. A *compactum* is a compact metric space. A *continuum* is a connected compactum. An *arc* is a continuum homeomorphic to the closed unit interval $[0,1] \subset \mathbb{R}$. A space X is *arcwise connected* if for every $a, b \in X$ there is an arc $A \subset X$ such that $a, b \in A$.

A compactum Y is a *compactification* of a space S with a space X if Y contains a subset $\widetilde{X}$ homeomorphic to X such that $Y \setminus \widetilde{X}$ is a dense subset of Y homeomorphic to S. We say that $\widetilde{X}$ is the *remainder* of the compactification. By saying that Y is a compactification of S with X without mentioning $\widetilde{X}$, we frequently assume that X and S are contained in Y in such a way that $S = Y \setminus X$ is dense in Y. In this context we simply say that X is the remainder of the compactification.

Properties of compactifications of the closed half-line $[0,\infty) \subset \mathbb{R}$ were studied in several continuum theory papers leading to many interesting results. See, for example, [1–3,11,20,24,25]. In this paper we replace the half-line with the plane and also with the closed half-plane, and study arcwise connected compactifications of these 2-dimensional objects.

For each non-separating 1-dimensional continuum X contained in the 2-dimensional sphere S^2, $S^2 \setminus X$ is homeomorphic to the plane. The sphere S^2 is, therefore, a compactification of the plane with X as the remainder. S^2 is locally connected. It follows that for every non-separating 1-dimensional continuum $X \subset \mathbb{R}^2$, even as locally complicated as the pseudoarc is, there exists a locally connected compactification of the plane with X as the remainder. In his continuum theory seminar at Auburn University, Professor Andrew Lelek asked the following question:

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Question 1.1 *(Lelek).* Which continua are remainders of locally connected compactifications of the plane?

The above question is still open. In this paper we study a similar problem with local connectedness replaced by arcwise connectedness. Of course, each locally connected continuum is arcwise connected. We give the following characterization of continua that are remainders of arcwise connected compactifications of the plane.

Theorem 1.2. *A continuum X is pointed* 1*-movable if and only if there is an arcwise connected compactification of the plane with X as the remainder.*

Movability, n-movability and movability in the pointed category were introduced by Karol Borsuk (see [4,6–8]). In the next section we are going to remind the reader of the standard definitions of these notions together with some properties that are useful in our proof of Theorem 1.2. However, we would like to start here with a characterization, due to Józef Krasinkiewicz [17, Theorem 4.2], that expresses pointed 1-movability in terms of the fundamental group. We do that with a dual purpose in mind. First, Krasinkiewicz's characterization is very useful in constructing continua that are not pointed 1-movable. Second, it is unexpected that pointed 1-movability, the notion that may be defined purely in algebraic terms, is characterized in the language of continuum theory by arcwise connected compactifications of the plane.

An inverse sequence of groups $\{G_n, f_{nm}\}$ is *Mittag-Leffler* if for each index n there is $n_0 \geqslant n$ such that $f_{nm}(G_{nm}) = f_{nn_0}(G_{n_0})$ for each $m \geqslant n_0$. For example, consider the dyadic solenoid Σ_2 expanded into the inverse sequence $\{X_n, f_{nm}\}$ where X_n is the unit circle S^1 in the complex plane and $f_{nm}(x) = x^{2^{m-n}}$ for each $x \in S^1$ and $m \geqslant n$. The corresponding sequence of the fundamental groups $\{\pi_1(S^1, 1), (f_{nm})_\#\}$ is the inverse sequence of copies of $\mathbb{Z}$ with $(f_{nm})_\#$ being the multiplication by 2^{m-n}. Thus, $(f_{nm})_\#(\pi_1(S^1, 1)) = 2^{m-n}\mathbb{Z}$ and the sequence of the fundamental groups defining Σ_2 is not Mittag-Leffler.

Theorem 1.3. *(See Krasinkiewicz [17, Theorem 4.2].) Suppose that a continuum X is homeomorphic to the inverse limit of an inverse sequence $\{(X_n, x_n), f_{nm}\}$ of pointed* ANR*'s. Then, X is pointed* 1*-movable if and only if the corresponding sequence of the fundamental groups $\{\pi_1(X_n, x_n), (f_{nm})_\#\}$ is Mittag-Leffler.*

Using the above theorem one may easily construct examples of continua that are not pointed 1-movable. For instance, all solenoids and the Case–Chamberlin continuum [10] are not pointed 1-movable. In particular, it follows from Theorem 1.2 that there is no arcwise connected compactification of the plane with a solenoid as the remainder.

In our proof of 1.2, we use a result of Krasinkiewicz and the author [18, Theorem 3.1] (quoted in the next section as Theorem 2.6) characterizing pointed 1-movable continua as continua in which every two points may be joined by approximative paths (see the next section for more details). A similar characterization was obtained by Kodama [14, Theorem 2] who used Čech paths defined by maps from the remainder of the Stone–Čech compactification of $[0, 1] \times [0, 1)$. Since a continuous image of an approximate path is an approximate path and a continuous image of a Čech path is Čech [18, Theorem 3.1] and [14, Theorem 2] gave alternate proofs of the theorem asserting that a continuous image of a pointed 1-movable continuum is also pointed 1-movable (see [17,21] for the original proofs of this theorem, see [13,15,16,26] for related earlier results). Theorem 1.2 yields yet another, even more elementary explanation why the class of pointed 1 movable continua is closed under continuous mappings. In fact, if Y is an arcwise connected compactification of the plane with X as the remainder and f is a map of X onto Z, then Y/f is an arcwise connected compactification of the plane with Z as the remainder. (Y/f is the quotient space resulting from the decomposition of Y into $f^{-1}(z)$ for $z \in Z$ and single points in $Y \setminus X$.)

Since each locally connected continuum is arcwise connected, it follows from 1.2 that if there is a locally connected compactification of the plane with X as the remainder, then X has to be a pointed 1-movable continuum.

Question 1.4. Is there a pointed 1-movable continuum X such that there is no locally connected compactification of the plane with X as the remainder?

The negative answer to the above question would provide an answer to the Lelek's question.

2. Pointed 1-movable continua

Let Q denote the Hilbert cube $[0,1]^\infty$.

Definition 2.1. Let $X \subset Q$ be a compactum and let $x_0 \in X$. We say that X is *movable* if for every neighborhood U of X in Q there is a neighborhood U_0 of X in U with the property that for each neighborhood V of X in Q there is a homotopy $h: U_0 \times [0,1] \to U$ such that $h(u,0) = u$ and $h(u,1) \in V$ for each $u \in U_0$. X is *pointed movable* if h can be chosen such that it additionally satisfies the condition $h(x_0, t) = x_0$ for each $t \in [0,1]$.

Movability and pointed movability of X do not depend on the embedding of X in Q, see [8, 1.4, p. 150 and 8.6, p. 167]. If X is a continuum, then pointed movability does not depend on the choice of x_0. Every plane continuum is pointed movable, see [8, 6.1, p. 160]. It is not known whether each movable continuum is pointed movable (see [21,17]).

Definition 2.2. Let X be a continuum contained in the Hilbert cube Q and let $x_0 \in X$. We say that X is *pointed* 1-*movable* if for every neighborhood U of X in Q there is a neighborhood U_0 of X in U such that for every 1-dimensional continuum $C \subset U_0$ containing x_0 and for each neighborhood V of X in Q there is a homotopy $h: C \times [0,1] \to U$ such that

(1) $h(c,0) = c$ and $h(c,1) \in V$ for each $c \in C$, and
(2) $h(x_0, t) = x_0$ for each $t \in [0,1]$.

We say that X is 1-*movable* if satisfies a similar definition without condition (2.2).

1-movability and pointed 1-movability do not depend of the embedding of X in Q and on the choice of x_0, see [8, 11.2, p. 171] and [17, 1.1 and 1.2]. The notions of movability, 1-movability and pointed 1-movability coincide for curves (1-dimensional continua), see [21,27]. This is not the case for higher dimensional continua. There are locally connected (and, thus, pointed 1-movable) continua in $\mathbb{R}^3$ that are not movable, see [5,22]. Also, there exists a 2-dimensional, 1-movable continuum which is not pointed 1-movable [12]. A movable and pointed 1-movable continuum is pointed movable [17, 1.4] (see also [9] for a related result).

In our proof of Theorem 1.2, we use the following theorem due to Krasinkiewicz [17].

Theorem 2.3. *(See [17, Theorem 1.1].) Suppose $X \subset Q$ is a pointed* 1*-movable continuum and U is a neighborhood of X in Q. Then there is a neighborhood U_0 of X in U such that for each path $\omega: [0,1] \to U_0$ with endpoints $\omega(0), \omega(1) \in X$ and each neighborhood V of X, ω is homotopic in U, relatively the endpoints, to a path in V.*

Corollary 2.4. *Suppose $X \subset Q$ is a pointed* 1*-movable continuum. There is a sequence $W_0, W_1, \ldots$ of open subsets of Q such that $Q = W_0 \supset W_1 \supset W_2 \supset \cdots$, $\bigcap_{n=0}^{\infty} \mathrm{cl}(W_n) = X$, and the following condition is satisfied*:

For each path $\omega: [0,1] \to Q$ with endpoints $\omega(0), \omega(1) \in X$ there is a homotopy $h: [0,1] \times [0,\infty) \to Q$ such that

(1) *$h(t,0) = \omega(t)$ for each $t \in [0,1]$,*
(2) *$h(0,s) = \omega(0)$ and $h(1,s) = \omega(1)$ for each $s \in [0,\infty)$, and*
(3) *$h([0,1] \times [i, i+1]) \subset W_i$ for each $i = 0, 1, \ldots$.*

Additionally, if $\omega([0,1]) \subset W_k$ for some positive integer k, then $h([0,1] \times [0,\infty)) \subset W_{k-1}$.

Proof. For each $i = 0, 1, \ldots$, let B_i denote the set of points in Q whose distance from X is less than 2^{-i}. We will construct $W_0, W_1, \ldots$ by induction. Set $W_0 = Q$. Suppose that $W_0, \ldots, W_{i-1}$ have been constructed for some positive integer i. Use Theorem 2.3 to get a neighborhood $W_i \subset W_{i-1} \cap B_i$ of X such that for each path $\alpha: [0,1] \to W_i$ with endpoints $\alpha(0), \alpha(1) \in X$ and each neighborhood V of X, α is homotopic in W_{i-1}, relatively the endpoints, to a path in V.

Let $\omega: [0,1] \to Q$ be a path with endpoints $\omega(0), \omega(1) \in X$. Suppose $\omega([0,1]) \setminus W_2 \neq \emptyset$. Since Q is contractible there is a homotopy $h: [0,1] \times [0,1] \to Q$ such that $h(t,0) = \omega(t)$ and $h(t,1) \in W_2$ for each $t \in [0,1]$, and $h(0,s)=\omega(0)$ and $h(1,s)=\omega(1)$ for each $s\in[0,1]$. Suppose that, for some positive integer i, the homotopy h has been defined on $[0,1]\times[0,i]$ such that $h(0,s)=\omega(0)$ and $h(1,s)=\omega(1)$ for each $s\in[0,i]$, and $h(t,i)\in W_{i+1}$ for each $t\in[0,1]$. By the choice of W_{i+1}, h can be extended to $[0,1]\times[0,i+1]$ such that $h(0,s)=\omega(0)$ and $h(1,s)=\omega(1)$ for each $s\in[i,i+1]$, $h([0,1]\times[i,i+1])\subset W_i$ and $h(t,i+1)\in W_{i+2}$ for each $t\in[0,1]$. This inductive step completes the construction of h in the case $\omega([0,1])\setminus W_2\neq\emptyset$. Now, suppose that $\omega([0,1])\subset W_2$. Let $j\geqslant 1$ be the greatest integer that $\omega([0,1])\subset W_{j+1}$. Set $h(t,s)=\omega(t)$ for each $t\in[0,1]$ and $s\in[0,j]$. Then, extend h by induction as it was done above over $[0,1]\times[i,i+1]$ for each $i\geqslant j$. □

The following definition was introduced in [18] in the language of inverse sequences of absolute neighborhood retracts ANR's.

Definition 2.5. Suppose $X\subset Q$ is a continuum and $a,b\in X$. We say that a and b are in the same *approximative path component* of X if there is a map $h:[0,1]\times[0,\infty)\to Q$ such that

- $h(0,s)=a$ and $h(1,s)=b$ for each $s\in[0,\infty)$, and
- for each neighborhood U of X in Q there is $s\geqslant 0$ such that

$$h\big([0,1]\times[s,\infty)\big)\subset U.$$

Any map h with the above properties is called an *approximative X-path* between a and b. In [17] points in the same approximative path component of X are called *joinable* in X.

Observe that h defined in Corollary 2.4 is an approximative path.

We use the following characterization of pointed 1-movability given by Krasinkiewicz and the author in [18]. The characterization was stated there in the language of inverse sequences. It is convenient for us to reformulate the characterization in the language of neighborhoods in the Hilbert cube. Even though the two versions are clearly equivalent, we include a brief proof of the following theorem for the sake of completeness.

Theorem 2.6. *(See [18, Theorem 3.1].) Suppose $X\subset Q$ is a continuum. Then the following conditions are equivalent*:

(1) *X is pointed 1-movable.*
(2) *Each two points in X are in same approximative path component of X.*
(3) *X has countably many approximative path components.*

Proof. The implication (1) ⇒ (2) follows from Corollary 2.4. (2) ⇒ (3) is obvious. We will prove (3) ⇒ (1) using [18, Theorem 3.1].

For each $i=0,1,\ldots$, consider the partition of $[0,1]$ into 2^i not overlapping closed subintervals of equal length. Let $\mathcal{I}_i$ denote the collection of these intervals. Let $\mathcal{C}_i$ be the collection of the cubes in the form $\prod_{j=0}^{i} I_i\times[0,1]^\infty$ where $I_j\in\mathcal{I}_i$ for $j=0,\ldots,i$. Let X_i denote the union of elements of $\mathcal{C}_i$ intersecting X. Observe that X_i is an ANR containing X in its interior such that $X_0=Q\supset X_1\supset X_2\supset\cdots$ and $\bigcap_{j=0}^{\infty}X_j=X$. Let f_{nm} denote the inclusion of $X_m\subset X_n$ for $n\leqslant m$. Clearly, X is homeomorphic to the inverse limit of the sequence $\underline{X}=\{X_n,f_{nm}\}$.

Suppose $h:[0,1]\times[0,\infty)\to Q$ is an approximative X-path in the sense of Definition 2.5. Then, for each integer $i=0,1,\ldots$, there is $s(i)>0$ such that $h([0,1]\times[s(i),\infty))\subset X_i$. Set $\omega_i(t)=h(t,s(i))$. Observe that $\{\omega_i\}_{i=0}^{\infty}$ is an approximative path in the sense of [18, B, p. 142]. It follows that X has countably many approximative paths components in the sense of [18, B, p. 142]. Now, the implication (3) ⇒ (1) follows from (iv) ⇒ (i) of [18, Theorem 3.1]. □

3. Main results

Definition 3.1. Suppose that X and Z are subsets of a certain space Y. We say that Z *can be pushed in Y toward X* if there is a homotopy $h:Z\times[0,\infty)\to Y$ such that

(1) $h(z,0)=z$ for each $z\in Z$,

(2) $h(x, s) = x$ for each $x \in Z \cap X$ and each $s \in [0, \infty)$, and
(3) for each neighborhood U of X in Y, there is $s \in [0, \infty)$ such that

$$h\big(Z \times [s, \infty)\big) \subset U.$$

The following proposition is an easy consequence of Definitions 2.5 and 3.1.

Proposition 3.2. *Suppose that X is a continuum contained in a space Y such that each arc $A \subset Y$ can be pushed toward X. Let $a, b \in X$ be two points in the same arc component of Y. Then a and b are in the same approximative path component of X.*

Proposition 3.3. *Suppose that X and Y are continua such that $X \subset Y$ and each component of $Y \setminus X$ is an open set homeomorphic to $\mathbb{R}^n$ for some $n \geqslant 2$ (n may vary for different components $Y \setminus X$). Then, each arc $A \subset Y$ can be pushed in Y toward X.*

Proof. Denote by $\mathbf{0}$ the origin $\mathbb{R}^n$. Let $r_n : (\mathbb{R}^n \setminus \{\mathbf{0}\}) \times [0, \infty) \to \mathbb{R}^n \setminus \{\mathbf{0}\}$ be defined by

$$r_n(z, s) = \begin{cases} \frac{sz}{|z|}, & \text{if } |z| < s; \\ z, & \text{if } |z| \geqslant s. \end{cases}$$

Let $\mathcal{K}$ denote the collection of components of $Y \setminus X$. Since each element of $\mathcal{K}$ is open in Y, $\mathcal{K}$ is countable.

For each $K \in \mathcal{K}$, let $n(K)$ be such that K is homeomorphic to $\mathbb{R}^{n(K)}$. There is a homeomorphism g_K of K onto $\mathbb{R}^{n(K)}$ such that $\mathbf{0} \notin g_K(A \cap K)$. Define $h : A \times [0, \infty) \to Y$ by the following

$$h(a, s) = \begin{cases} {g_K}^{-1} \circ r_{n(K)}(g_K(a), s), & \text{if } a \in K \text{ for some } K \in \mathcal{K}; \\ a, & \text{if } a \in X. \end{cases}$$

Observe that h satisfies Definition 3.1. □

Corollary 3.4. *Suppose Y is a compactification of the plane with X as the remainder. Then the each arc $A \subset Y$ can be pushed in Y toward X.*

Corollary 3.5. *Suppose Y is a compactification of the plane with a continuum X as the remainder. If Y has countably many arc components, then X is pointed* 1*-movable.*

Proof. By Corollaries 3.4 and 3.2, X has countably many approximative path components. Now, the corollary follows from Theorem 2.6. □

Lemma 3.6. *Suppose that X and $W_0, W_1, \ldots$ are as in Corollary* 2.4. *Let M be a closed topological disk contained in the plane. Denote by T the interior of M. Suppose that L is an arc contained in the boundary of M. Let a and b denote the endpoints of L. Set $L^\circ = L \setminus \{a, b\}$ and $T^* = T \cup L^\circ$. Suppose $q : L \to Q$ is a map such that $q(a), q(b) \in X$. Then, there a map $g : T^* \to Q$ such that*

(1) *$g(x) = q(x)$ for each $x \in L^\circ$,*
(2) *for every neighborhood U of X, there exists a compact set $K \subset T^*$ such that $g(T^* \setminus K) \subset U$, and*
(3) *if $q(L) \subset W_k$ for some positive integer k, then $g(T^*) \subset W_{k-1}$.*

Proof. Let φ be a homeomorphism of $T \cup L$ onto $(0, 1) \times (0, \infty) \cup [0, 1] \times \{0\}$. Clearly, φ restricted to L is a homeomorphism onto $[0, 1] \times \{0\}$. Let p be a homeomorphism of $[0, 1]$ onto L such that $\varphi \circ p(t) = (t, 0)$ for each $t \in [0, 1]$. Let ω denote the composition $q \circ p$. Observe that $\omega(0), \omega(1) \in X$. Let $h : [0, 1] \times [0, \infty) \to Q$ be as in corollary 2.4. Define g as $h \circ \varphi$ restricted to T^*. We will check that condition (2) is satisfied and leave proofs of (1) and (3) to the reader.

Let U be a neighborhood U of X. Since $\bigcap_{n=0}^{\infty} \mathrm{cl}(W_n) = X$, there an index i such that $W_i \subset U$. By Corollary 2.4(3), $h([0, 1] \times [i, \infty)) \subset W_i$. Since $h(\{0, 1\} \times [0, i]) \subset X$, there are numbers c and d such that $0 < c < d < 1$, $h([0, c] \times [0, i]) \subset U$ and $h([d, 1] \times [0, i]) \subset U$. Now, set $K = q^{-1}([c, d] \times [0, i])$ and observe that (2) is satisfied. □

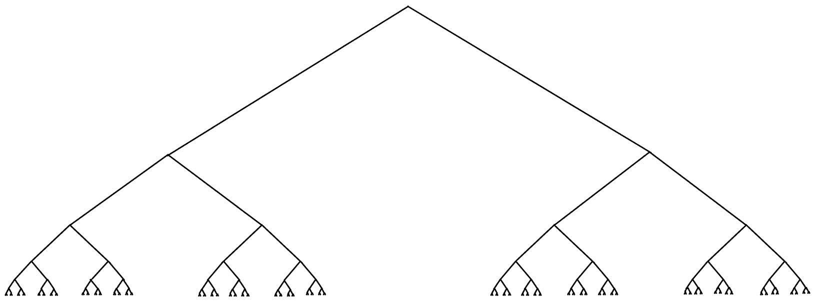

Fig. 1.

Theorem 3.7. *Suppose X is a pointed 1-movable continuum. Let $\mathbb{R}^2_+$ denote the closed half-plane $\{(x, y) \in \mathbb{R}^2 \mid y \geqslant 0\}$. Then there is an arcwise connected compactification of $\mathbb{R}^2_+$ with X as the remainder.*

Proof. Consider the square $[0, 1]^2$. Denote by J the segment $[0, 1] \times \{0\}$ with the endpoints $a = (0, 0)$ and $b = (1, 0)$. Let D be the dendrite in Fig. 1 (see [23, 10.39]). We assume that $D \subset [0, 1]^2$ is such that the intersection of J and D is the Cantor set C in the base of D with a and b in C as in the figure. For any two points $c, d \in D$ let $\langle c, d\rangle_D$ denote the arc in D between c and d. Let $L = \langle a, b\rangle_D$. Define T to be the component of $[0, 1]^2 \setminus (L \cup J)$ whose closure contains J. Set $L^\circ = L \setminus \{a, b\}$ and $T^* = T \cup L^\circ$. Since $\mathbb{R}^2_+$ and T^* are homeomorphic, to prove the theorem it will be enough to show an arcwise connected compactification of T^* with X as the remainder.

Arrange all components of $T \setminus D$ into a non-repetitive sequence $(T_1, T_2, \ldots)$. Observe that the closure of each T_i intersects C at exactly two points, say a_i and b_i. Set $L_i = \langle a_i, b_i\rangle_D$, $L_i^\circ = L_i \setminus \{a_i, b_i\}$ and $T_i^* = T_i \cup L_i^\circ$. Observe that L_i° is the boundary of T_i in T. For each point $x \in T_i^*$, let $\delta_i(x)$ denote the distance of x from the boundary of T_i in $[0, 1]^2$.

There is a map f of C onto X, see [19, §41, VI, Corollary 3a]. Let $Z = D/f$ be the quotient space resulting from the decomposition of D into $f^{-1}(x)$ for $x \in X$ and single points. Z is a locally connected continuum as a continuous image of a dendrite. Let q denote the quotient map of D onto Z. For each $x \in X$, denote by $\varphi(x)$ the point $q(f^{-1}(x))$. Observe that φ is a homeomorphism of X onto $q(C)$. To simplify the notation, we assume that $X \subset Z$ by treating $\varphi(x)$ as x. In this convention, q restricted to $D \setminus C$ is a homeomorphism onto $Z \setminus X$ and

(1) $X = \mathrm{cl}(q(D \setminus C)) \setminus q(D \setminus C)$.

We also assume that Z is embedded in the Hilbert cube Q. It follows that $X \subset Z \subset Q$. Let $W_0, W_1, \ldots$ be as in Lemma 3.6. For each positive integer i, let $k(i)$ be the greatest integer such that $q(L_i) \subset W_{k(i)}$. Observe that $\lim_{i\to\infty} k(i) = \infty$ since $\lim_{i\to\infty} \mathrm{diam}(L_i) = 0$. Using Lemma 3.6 we get an extension of q restricted to L_i° to a map $g_i : T_i^* \to Q$ such that

(2) for every neighborhood U of X in Q, there exists a compact set $K_i \subset T_i^*$ such that $g_i(T_i^* \setminus K_i) \subset U$, and
(3) $g_i(T_i^*) \subset W_{k(i)-1}$ if $k(i) > 0$.

Let g be the union $\bigcup_{i=1}^\infty g_i$. Observe that $g : T^* \to Q$ is continuous and

(4) $g(x) = q(x)$ for each $x \in D \setminus C$.

Let U be a neighborhood of X in Q. Since $\bigcap_{n=0}^\infty \mathrm{cl}(W_n) = X$, there an index n such that $W_n \subset U$. There is an integer $m \geqslant 0$ such that $k(i) > n$ for each $i > m$. It follows that $\bigcup_{i>m} g_i(T_i^*) \subset W_n \subset U$. Let K_i be as in (2). Define $K = \bigcup_{i=0}^m K_i$ and observe that $g(T^* \setminus K) \subset U$. So, we proved that

(5) for every neighborhood U of X in Q there exists a compact set $K \subset T^*$ such that $g(T^* \setminus K) \subset U$.

Denote by Q' the product $Q \times [0,1]^\infty$. Let $p : Z \to Q'$ be such that $p(z) = (z, 0, 0, 0, \ldots)$. For each positive integer i and each point $x \in T_i^*$, let $e_i(x) = (y, y_1, y_2, \ldots)$ be such that

(i) $y = g(x)$,
(ii) $y_m = 0$ unless $m \in \{3i-2, 3i-1, 3i\}$,
(iii) $y_{3i-2} = \delta_i(x)$, and
(iv) $(y_{3i-1}, y_{3i}) = x\delta_i(x)$.

Since $g(x) = g_i(x) = q(x)$ (see (4)) and $\delta_i(x) = 0$ for $x \in L_i^\circ$,

(6) e_i and $p \circ q$ agree on L_i°.

It follows that $e_i(L_i^\circ) \cap p(X) = \emptyset$. Since $\delta_i(x) > 0$ for $x \in T_i$,

(7) $e_i(T_i^*) \cap p(X) = \emptyset$.

It follows from (5) and the definition of δ_i that

(8) $\mathrm{cl}(e_i(T_i^*)) \setminus e_i(T_i^*) \subset p(X)$.

Conditions (iii) and (iv) guarantee that e_i is one-to-one on T_i. Since $\delta_i(x) > 0$ for $x \in T_i$, $\delta_i(x) = 0$ for $x \in L_i^\circ$, $e_i(T_i) \cap e_i(L_i^\circ) = \emptyset$. Since e_i is one-to-one on L_i° (by (6)), e_i is one-to-one on T_i^*. Since T_i^* is locally compact, it follows from (7) and (8) that

(9) e_i is an embedding of T_i^* into Q'.

Conditions (ii) and (iii) guarantee that

(10) $e_i(T_i) \cap e_j(T_j) = \emptyset$ for $j \neq i$.

Let e denote the union $\bigcup_{i=1}^\infty e_i$. By (6)–(10), $e : T^* \to Q'$ is an embedding such that $e(T^*) \cap p(X) = \emptyset$ and $\mathrm{cl}(e(T^*)) \setminus e(T^*) \subset p(X)$. By (1) and (6), $\mathrm{cl}(e(T^*)) \setminus e(T^*) = p(X)$. Set $E = \mathrm{cl}(e(T^*))$. Observe that E is a compactification of T^* with the remainder $p(X)$ homeomorphic to X. By (6), $e(d) = p(q(d))$ for each $d \in D \setminus C$. It follows that $p(q(D \setminus C)) \subset E$. Since $Z = q(D \setminus C) \cup X$ and $p(X) \subset E$, $p(Z) \subset E$. Since Z is locally connected, it is arcwise connected. So, both $p(Z)$ and $e(T^*)$ are arcwise connected. Since $p(Z)$ and $e(T^*)$ intersect and E is their union, E is arcwise connected. $\square$

Theorem 3.8. *Suppose X is a pointed 1-movable continuum. Then there is an arcwise connected compactification of the plane with X as the remainder.*

Proof. By Theorem 3.7, there is a compactification E of the upper half-plane $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \mid y \geqslant 0\}$ with X as the remainder. Thus we may assume that $X \subset E$ and $E \setminus X = \mathbb{R}^2_+$. Let $R = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$. In our convention $R \subset E$. To complete the proof of the theorem, consider $Y = E \times \{0, 1\}$. For any $(u, i), (v, j) \in Y$, set $(u, i) \sim (v, j)$ if either $(u, i) = (v, j)$ or $u = v \in X \cup R$. Observe that $Y/\!\sim$ is an arcwise connected compactification of the plane with remainder X. $\square$

Theorem 3.9. *Suppose X is a continuum. Then the following conditions are equivalent.*

(1) *X is pointed 1-movable.*
(2) *There is an arcwise connected compactification of the plane with X as the remainder.*
(3) *X can be embedded in a space Y such that*
- *each arc $A \subset Y$ can be pushed in Y toward X, and*
- *X is contained in the union of countably many arc components of Y.*

Proof. The implication (1) ⇒ (2) was proven in Theorem 3.8. (2) ⇒ (3) follows from Corollary 3.4. (3) ⇒ (1) follows from Proposition 3.2 and Theorem 2.6. □

Observe that Theorem 1.2, whose proof we promised in the introduction, is a part of Theorem 3.9.

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